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Theoretical Limits of Damping Attainable by Smart Beams with Rate Feedback

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Theoretical Limits of Damping Attainable by Smart Beams with Rate Feedback

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ABSTRACT

Using a generally accepted model we present a comprehensive analysis (within the page limitation) of an Euler-Bernoulli beam with PZT sensor-actuator and pure rate feedback. The emphasis is on the root locus — the dependence of the attainable damping on the feedback gain. There is a critical value of the gain beyond which the damping decreases to zero. We construct the time-domain response using semigroup theory, and show that the eigenfunctions form a Riesz basis, leading to a "modal" expansion.

1. INTRODUCTION

In this paper we present a comprehensive analysis of an Euler-Bernoulli beam with PZT sensor-actuator along its entire length. The sensor output is a charge in a condenser and the actuator input is the current, a differentiator circuit being then an essential component, yielding "rate feedback." We use a generally accepted model.[1-4] Tzou *et al.*[4] present purely computational results and seem to be unaware of a purely theoretical analysis given earlier by Chen *et al.*[5] The most important design parameter is the control gain and the damping attainable — we construct a full root-locus analysis (omitting details to keep within the page limitation). We also unearth a curious phenomenon — the existence of a deadbeat mode (real eigenvalue) not noticed hitherto. We show that the eigenvalues are the roots of an entire function of order one-half, proving in particular the existence of a countably infinite number of eigenvalues. We also show that the eigenfunctions form a Riesz basis. We also construct the Green's function for the nonhomogeneous eigenvalue problem. As in Chen *et al.* we use the theory of semigroups of operators to obtain the time-domain solution. Our proof of the exponential stability is different from that in Chen *et al.*, as is our choice of the function space. We note that a similar analysis for a Timoshenko model (a "smart string") is given in Balakrishnan,[6] where there is a critical value of the gain at which there are no eigenvalues and the semigroup is actually nilpotent ("disappearing" solution).

2. MAIN RESULTS

The Euler-Bernoulli model formulates as

$$\left.\begin{array}{c} cf''''(t,s) + m\ddot{f}(t,s) = 0, \quad 0 < s < L,\ 0 < t \\ f(t,0) = 0 = f'(t,0); \quad f'''(t,L) = 0 \\ cf''(t,L) + \alpha\dot{f}'(t,L) = 0 \end{array}\right\} \tag{1}$$

where $f(t,s)$ is the displacement and the superdots indicate derivative with respect to t and the primes indicate derivative with respect to s. It is convenient to set

$$\nu^2 = \frac{m}{c}\,.$$

For a precise formulation of the time-domain response we need to specify first the choice of function spaces. We pick $L_2[0,L]$ for $f(t,\cdot)$. Let A_o denote the operator defined by

$$A_o f = cf''''$$

where

$$\mathcal{D}(A_o) = [f \mid f', f'', f''', f'''' \in L_2[0,L];\ f(0) = 0 = f'(0) = f'''(L)]\,.$$

Let

$$\mathcal{H} = L_2[0, L] \times E^1.$$

Define the operator A with domain and range in $\mathcal{H}$ by:

$$x = \left| \begin{matrix} f \\ b \end{matrix} \right|, \quad Ax = \left| \begin{matrix} A_o f \\ cf''(L) \end{matrix} \right|;$$

with domain

$$\mathcal{D}(A) = \left[\left| \begin{matrix} f \\ b \end{matrix} \right|, \; f \in \mathcal{D}(A_o) \text{ and } b = f'(L) \right].$$

It is convenient to adopt the notation

$$A_b x = cf''(L), \quad x \in \mathcal{D}(A).$$

Then for x in $\mathcal{D}(A)$:

$$\begin{aligned} [Ax, x] &= \int_0^L cf''''(s)\, \overline{f(s)}\, ds \; + \; cf''(L)\, \overline{f'(L)} \\ &= c \int_0^L |f''(s)|^2\, ds. \end{aligned}$$

It is readily seen that A has dense domain and is self-adjoint and nonnegative definite, and has compact resolvent. Also zero is not an eigenvalue. Let $\sqrt{A}$ denote the positive square root. On the product space

$$\mathcal{D}\left(\sqrt{A}\right) \times L_2[0, L]$$

introduce the "energy" inner product

$$[Y, Z]_E \; = \; \left[\sqrt{A}\, y_1, \sqrt{A}\, z_1\right] \; + \; m\, [y_2, z_2]$$

$$Y \; = \; \left| \begin{matrix} y_1 \\ y_2 \end{matrix} \right|, \quad Z \; = \; \left| \begin{matrix} z_1 \\ z_2 \end{matrix} \right|.$$

$$\mathcal{D}\left(\sqrt{A}\right) \; = \; \left[\left| \begin{matrix} f \\ b \end{matrix} \right| \; \middle| \; f'' \in L_2[0, L] \text{ and } b = f'(L), \; f(0) = f'(0) = 0 \right].$$

For y_1 in $\mathcal{D}(A)$, we see that

$$[Y, Y]_E = [Ay_1, y_1] + m\, [y_2, y_2] \; \sim \; \text{“energy” (potential + kinetic)}.$$

We denote the product space under this inner product by $\mathcal{H}_E$ and note that it is a Hilbert space. Let $\mathcal{A}$ denote the operator defined by:

$$\mathcal{A}Y \; = \; \left| \begin{matrix} f_2 \\ \frac{-cf_1''(L)}{\alpha} \\ \frac{-A_o f_1}{m} \end{matrix} \right|, \qquad Y \; = \; \left| \begin{matrix} x \\ f_2 \end{matrix} \right| \; = \; \left| \begin{matrix} f_1(\cdot) \\ f_1'(L) \\ f_2(\cdot) \end{matrix} \right|$$

and

$$\mathcal{D}(\mathcal{A}) \; = \; \left| x = \left| \begin{matrix} f_1(\cdot) \\ f_1'(L) \end{matrix} \right| \in \mathcal{D}(A) \right|, \qquad \left| \begin{matrix} f_2(\cdot) \\ f_1''(L) \end{matrix} \right| \in \mathcal{D}\left(\sqrt{A}\right).$$

Thus defined we can verify that

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*)$$

and that $\mathcal{A}$ is dissipative:

$$\tfrac{1}{2}[(\mathcal{A} + \mathcal{A}^*)Y, \, Y] \; = \; \mathrm{Re}\, [\mathcal{A}Y, Y]_E \; = \; \tfrac{-1}{\alpha} ||A_b x||^2 \; = \; \tfrac{-1}{\alpha}\, c^2\, |f_1''(L)|^2.$$

It is readily verified that $\mathcal{A}$ has a compact resolvent and that $\mathcal{A}$ generates a C_0 contraction semigroup. With these definitions, the system (1) goes over into the abstract formulation:

$$\dot{Y}(t) = \mathcal{A}Y(t). \tag{2}$$

This choice of the function space is technically different from that in Chen *et al.*[5]

Eigenvalues and eigenfunctions of $\mathcal{A}$

Our primary interest is in the modal decomposition — the eigenvalues of $\mathcal{A}$ and the corresponding eigenfunctions. Or, equivalently, in the resolvent of $\mathcal{A}$. Let $\mathcal{R}(\lambda, \mathcal{A})$ denote the resolvent of $\mathcal{A}$. Let

$$\mathcal{R}(\lambda, \mathcal{A})Y = Z$$

where

$$Y = \begin{vmatrix} h_1 \\ b \\ h_2 \end{vmatrix}.$$

Since $Z \in \mathcal{D}(\mathcal{A})$, we can write

$$Z = \begin{vmatrix} f_1(\cdot) \\ f_1'(L) \\ f_2(\cdot) \end{vmatrix}$$

and

$$(\lambda I - \mathcal{A})Z = Y$$

yields

$$\begin{aligned} \lambda f_1 - f_2 &= h_1 \\ \lambda f_2 + \frac{A_o f_1}{m} &= h_2 \\ \lambda f_1'(L) + \frac{c f_1''(L)}{\alpha} &= b. \end{aligned}$$

Hence

$$\left.\begin{aligned} \lambda^2\nu^2 f_1(s) + f_1''''(s) &= \nu^2(h_2(s) + \lambda h_1(s)), \quad 0 < s < L \\ \lambda\alpha f_1'(L) + c f_1''(L) &= \alpha b \\ f_1(0) \;=\; 0 &= f_1'(0) \;=\; f_1'''(L). \end{aligned}\right\} \tag{3}$$

Eigenvalues

First we consider the eigenvalue problem, setting

$$h_1 = 0 = h_2; \quad b = 0.$$

Let

$$\gamma = \sqrt{\lambda\nu}\, e^{i\theta/2} e^{i\pi/4}, \quad \gamma^4 = -\lambda^2\nu^2$$

where

$$\lambda = |\lambda| e^{i\theta}.$$

Then the solution satisfying the conditions at zero yields:

$$f_1(s) = a(\operatorname{Cosh}\gamma s - \operatorname{Cos}\gamma s) + b(\operatorname{Sinh}\gamma s - \operatorname{Sin}\gamma s), \quad 0 < s < L.$$

The constants a and b are then determined by the conditions at L:

$$a(\lambda\alpha\gamma(\operatorname{Sinh}\gamma L + \operatorname{Sin}\gamma L) + c\gamma^2(\operatorname{Cosh}\gamma L + \operatorname{Cos}\gamma L)) + b(\lambda\alpha\gamma(\operatorname{Cosh}\gamma L - \operatorname{Cos}\gamma L) + c\gamma^2(\operatorname{Sinh}\gamma L + \operatorname{Sin}\gamma L)) = 0$$

$$a\gamma^3(\operatorname{Sinh}\gamma L - \operatorname{Sin}\gamma L) + b\gamma^3(\operatorname{Cosh}\gamma L + \operatorname{Cos}\gamma L) = 0.$$

Let

$$H(\lambda) = \begin{vmatrix} \lambda\alpha\gamma(\mathrm{Sinh}\,\gamma L + \mathrm{Sin}\,\gamma L) & \lambda\alpha\gamma(\mathrm{Cosh}\,\gamma L - \mathrm{Cos}\,\gamma L) \\ + c\gamma^2(\mathrm{Cosh}\,\gamma L + \mathrm{Cos}\,\gamma L) & + c\gamma^2(\mathrm{Sinh}\,\gamma L + \mathrm{Sin}\,\gamma L) \\ \gamma^3(\mathrm{Sinh}\,\gamma L - \mathrm{Sin}\,\gamma L) & \gamma^3(\mathrm{Cosh}\,\gamma L + \mathrm{Cos}\,\gamma L) \end{vmatrix} \tag{4}$$

and

$$D(\lambda) = \mathrm{Det}\, H(\lambda).$$

Then

$$\begin{aligned} D(\lambda) &= (\gamma^4)\,[(\mathrm{Cosh}\,\gamma L + \mathrm{Cos}\,\gamma L)(\lambda\alpha(\mathrm{Sinh}\,\gamma L + \mathrm{Sin}\,\gamma L) + c\gamma(\mathrm{Cosh}\,\gamma L + \mathrm{Cos}\,\gamma L)) \\ &\quad - (\mathrm{Sinh}\,\gamma L - \mathrm{Sin}\,\gamma L)(\lambda\alpha(\mathrm{Cosh}\,\gamma L - \mathrm{Cos}\,\gamma L) + c\gamma(\mathrm{Sinh}\,\gamma L + \mathrm{Sin}\,\gamma L))] \\ &= 2\gamma^4\,[c\gamma(1 + \mathrm{Cosh}\,\gamma L\,\mathrm{Cos}\,\gamma L) + \lambda\alpha(\mathrm{Sinh}\,\gamma L\,\mathrm{Cos}\,\gamma L + \mathrm{Cosh}\,\gamma L\,\mathrm{Sin}\,\gamma L)]\,. \end{aligned} \tag{5}$$

We note that zero is not an eigenvalue. The eigenvalues $\{\lambda_k\}$ are thus determined by the nonzero roots of

$$c\gamma(1 + \mathrm{Cosh}\,\gamma L\,\mathrm{Cos}\,\gamma L) + \lambda\alpha\,(\mathrm{Sinh}\,\gamma L\,\mathrm{Cos}\,\gamma L + \mathrm{Cosh}\,\gamma L\,\mathrm{Sin}\,\gamma L) = 0$$

Or, using

$$\lambda = \frac{-i\gamma^2}{\nu}$$

we have

$$(1 + \mathrm{Cosh}\,\gamma L\,\mathrm{Cos}\,\gamma L) - \frac{i\gamma\alpha}{\nu c}(\mathrm{Sinh}\,\gamma L\,\mathrm{Cos}\,\gamma L + \mathrm{Cosh}\,\gamma L\,\mathrm{Sin}\,\gamma L) = 0. \tag{6}$$

Theorem 2.1[†]
$\mathcal{A}$ has exactly one real-valued eigenvalue.

Proof
Setting $L = 1$, and using α to denote $\frac{\alpha}{c\nu}$, and expressing the trigonometric products in (6) as sums, we have

$$\begin{aligned} f &= 1 + \mathrm{Cosh}\,\gamma\,\mathrm{Cos}\,\gamma - i\alpha\gamma(\mathrm{Sinh}\,\gamma\,\mathrm{Cos}\,\gamma + \mathrm{Cosh}\,\gamma\,\mathrm{Sin}\,\gamma) \\ f &= 1 + \tfrac{1}{2}(\mathrm{Cos}\,\gamma(1+i) + \mathrm{Cos}\,\gamma(1-i)) \\ &\quad - i\alpha\gamma\tfrac{1}{2}[\mathrm{Sinh}\,\gamma(1+i) + \mathrm{Sin}\,\gamma(1+i) + \mathrm{Sinh}\,\gamma(1-i) + \mathrm{Sin}\,\gamma(1-i)]. \end{aligned}$$

Hence making the 1:1 transformation

$$\gamma = x(i-1)$$

we obtain

$$f(\gamma) = g(x) = 1 + \tfrac{1}{2}(\mathrm{Cos}\,2x + \mathrm{Cosh}\,2x) - \alpha x(\mathrm{Sin}\,2x + \mathrm{Sinh}\,2x) \tag{7}$$

yielding an equivalent expression for determining the eigenvalues. Note that $g(\cdot)$ is real-valued for real values of x. Further

$$g(0) = 2$$

while, as $x \to \infty$, (x real), we note that

$$g(x) \to -\infty.$$

Hence there is a positive real root. Denote it x_0. Then

$$\lambda = -\gamma^2 i = -x^2(i-1)^2 i = -2x^2.$$

Hence

$$\lambda_0 = -2x_0^2$$

† Due to J. Lin; private communication.

is an eigenvalue. We note that x_1 is the ony real-valued root of $g(\cdot)$. Indeed, if there is a real-valued eigenvalue of $\mathcal{A}$, we must have, denoting it by λ_1,

$$\lambda_1 = -2x_1^2$$

and x_0 must be a root of $g(\cdot)$. Hence

$$x_0 = x_1.$$

Or, λ_0 is the only real-valued eigenvalue of $\mathcal{A}$.

We note that the corresponding eigenfunction is given by

$$\phi_1(s) = (\operatorname{Cosh}\gamma_0 - \operatorname{Cos}\gamma_0)(\operatorname{Cosh}\gamma_0 s - \operatorname{Cos}\gamma_0 s) - (\operatorname{Sinh}\gamma_0 - \operatorname{Sin}\gamma_0)(\operatorname{Sinh}\gamma_0 s - \operatorname{Sin}\gamma_0 s)$$

where

$$\gamma_0 = x_0(i-1), \quad \lambda = -2x_0^2.$$

Theorem 2.2 (Chen, et al.[5])
Let $\{\lambda_k\}$ denote the eigenvalues, and assume that

$$|\lambda_k| \to \infty.$$

Then

$$\lim_k \operatorname{Re}\lambda_k = \frac{-c}{L\alpha}. \tag{8}$$

Proof
See Chen *et al.*[5] for a proof.

The authors of Chen *et al.* however do not appear to offer a proof of the fact that the eigenvalues $\{\lambda_k\}$ are nonfinite in number. The fact that the resolvent is compact is not adequate to establish this; the compactness only assures that if nonfinite in number then $\{\lambda_k\}$ can be arranged so that

$$|\lambda_{k+1}| \geq |\lambda_k|$$

and

$$|\lambda_k| \to \infty \quad \text{as } k \to \infty.$$

For proving the fact that eigenvalues are denumerably infinite we can indicate a general technique.

Theorem 2.3
The eigenvalues $\{\lambda_k\}$ are denumerably infinite and such that

$$\sum_1^\infty \left|\operatorname{Im}\left(\frac{1}{\lambda_k}\right)\right| < \infty. \tag{9}$$

Proof
From (6) we see that for each α, the eigenvalues are the zeros of the function

$$d(\lambda) = (1 + \operatorname{Cosh}\gamma L \operatorname{Cos}\gamma L) - i\left(\frac{\alpha}{c\nu}\right)\gamma(\operatorname{Sinh}\gamma L \operatorname{Cos}\gamma L + \operatorname{Cosh}\gamma L \operatorname{Sin}\gamma L). \tag{10}$$

As a power series expansion will show, this is an entire function of the complex variable λ. Moreover it is of exponential type, of order $\frac{1}{2}$, and of completely regular growth. Further we can calculate that

$$h(\theta) = \overline{\lim_{r\to\infty}} \log|d(re^{i\theta})| = \sqrt{2}\max\left(|\operatorname{Sin}\tfrac{\theta}{2}|, |\operatorname{Cos}\tfrac{\theta}{2}|\right).$$

Let $n(r)$ denote the number of zeros of $d(\cdot)$ in the circle of radius r centered at zero. Then by the theorem of R.P. Boas (see Levin[7]) we have:

$$\lim_{r\to\infty} \frac{n(r)}{r^{1/2}} = \frac{1}{4\pi}\int_0^{2\pi} h(\theta)\, d\theta > 0.$$

Hence

$$\lim_{r\to\infty} n(r) = \infty,$$

or, the number of zeros is not finite. Moreover the function is of class A (see Levin[7] for the definition) since

$$\sup_{R>0} \int_0^R \frac{\log|d(s)\,d(-s)|}{1+s^2}\, ds < M_d < \infty.$$

The result (9) is a consequence. Q.E.D.

Remark

Applying Jensen's Theorem we have

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d'(re^{i\theta})}{d(re^{i\theta})}\, re^{i\theta}\, d\theta = n(r).$$

We can acutally compute this as a quick means of locating eigenvalues. There is a jump of 2 corresponding to each eigenvalue and its conjugate. This is shown in Figure 1 for

$$\frac{\alpha}{Lc\nu} = .01.$$

Eigenfunctions

The eigenfunction corresponding to the eigenvalue λ_k is given by

$$\mathbf{\Phi}_k = A_k \begin{vmatrix} \phi_k \\ \phi_k'(L) \\ \lambda_k \phi_k \end{vmatrix}$$

where

$$\phi_k(s) = c_k(\operatorname{Cosh}\gamma_k s - \operatorname{Cos}\gamma_k s) + d_k(\operatorname{Sinh}\gamma_k s - \operatorname{Sin}\gamma_k s)$$

where

$$H(\lambda_k)\begin{vmatrix} c_k \\ d_k \end{vmatrix} = 0$$

or, we may take

$$c_k = (\operatorname{Cosh}\gamma_k L - \operatorname{Cos}\gamma_k L); \quad d_k = -(\operatorname{Sinh}\gamma_k L - \operatorname{Sin}\gamma_k L)$$

or,

$$\phi_k(s) = A_k[(\operatorname{Cosh}\gamma_k L - \operatorname{Cos}\gamma_k L)(\operatorname{Cosh}\gamma_k s - \operatorname{Cos}\gamma_k s) - (\operatorname{Sinh}\gamma_k L - \operatorname{Sin}\gamma_k L)(\operatorname{Sinh}\gamma_k s - \operatorname{Sin}\gamma_k s)]. \quad (11)$$

Correspondingly:

$$\phi_k'(L) = 2A_k\gamma_k(\operatorname{Cosh}\gamma_k L - \operatorname{Cos}\gamma_k L)\operatorname{Sin}\gamma_k L.$$

The coefficient A_k may be chosen for appropriate normalization. For example we may make

$$||\mathbf{\Phi}_k|| = 1.$$

Note that $\overline{\lambda}_k$ is an eigenvalue of $\mathcal{A}^*$ and the corresponding eigenvector is:

$$\Psi_k = B_k \begin{vmatrix} \overline{\phi}_k(\cdot) \\ \overline{\phi}_k'(L) \\ -\overline{\lambda}_k \overline{\phi}_k(\cdot) \end{vmatrix}$$

where B_k is again a "normalization" scalar. Note that

$$\begin{aligned}
[\Phi_k, \Psi_k]_E &= \left(c \int_0^L \phi_k''(s)^2 \, ds \; - \; m\lambda_k^2 \int_0^L \phi_k(s)^2 \, ds \right) A_k \overline{B}_k \\
&= 4cA_k\overline{B}_k\gamma_k^4 c_k d_k \int_0^L (\text{Cosh}\, \gamma_n s \; \text{Cos}\, \gamma_n s \; + \; \text{Sinh}\, \gamma_n s \; \text{Sin}\, \gamma_n s) \, ds \\
&= 4cc_k d_k A_k \overline{B}_k \gamma_k \; \text{Cosh}\, \gamma_k L \; \text{Sin}\, \gamma_k L \\
&\neq 0.
\end{aligned}$$

In particular we may choose A_k, B_k so that

$$[\Phi_k, \Psi_k]_E = 1. \tag{12}$$

Further using a result of Gohberg and Krein[10] (we omit the details) we can establish that $\{\Phi_k, \Psi_k\}$ with the normalization (12) actually yield a Riesz basis for $\mathcal{H}_E$. In terms of this basis we have the ("modal") expansion for the solution of (2)

$$Y(t) \; = \; \sum_1^\infty a_k e^{\lambda_k t} \Phi_k \tag{13}$$

where

$$a_k \; = \; [Y(0), \Psi_k]_E$$

and as an easy byproduct, using (8), we see that the semigroup generated by $\mathcal{A}$ is exponentially stable (established in Chen *et al.* by different arguments).

Root Locus

Let us consider how the eigenvalues behave as the gain α is varied. For this purpose it is convenient to define

$$d(\lambda; \alpha) \; = \; M(\lambda) \; + \; \frac{\alpha}{c\nu} N(\lambda)$$

where

$$M(\lambda) \; = \; 1 \; + \; \text{Cosh}\, \gamma L \; \text{Cos}\, \gamma L$$

$$N(\lambda) \; = \; \frac{-i\gamma}{c\nu} (\text{Sinh}\, \gamma L \; \text{Cos}\, \gamma L \; + \; \text{Cosh}\, \gamma L \; \text{Sin}\, \gamma L).$$

Because of the analytic dependence of $d(\lambda; \alpha)$ on α, we can invoke the theory of algebraic or algebroidal functions[8,9] and note that

$$d(\lambda(\alpha); \alpha) \; = \; 0$$

will define $\lambda(\alpha)$ as a multivalued analytic function of α with isolated singularities, if any. In particular this allows us to define the sequence $\{\lambda_k(\alpha)\}$, $k = 1, 2, \ldots$ such that

$$\lambda_k(0) \; = \; \frac{i\mu_k^2}{L^2\nu}, \quad \mu_k \; = \; (2k-1)\frac{\pi}{2} \; + \; \varepsilon_k$$

(the "clamped-free" beam modes) and

$$\lim_{\alpha \to \infty} \lambda_k(\alpha) \; = \; \frac{i(k\pi - \varepsilon' k)^2}{L^2\nu}$$

("clamped-rolling" modes) and the real root

$$\lambda_0(\alpha)$$

is such that

$$\lim_{\alpha \to \infty} \lambda_0(\alpha) \; = \; 0, \quad \lim_{\alpha \to 0} \lambda_0(\alpha) \; = \; -\infty.$$

A plot of the locus of the real root is shown in Figure 2. Moreover

$$\lambda_k'(\alpha) \; = \; \frac{-1}{c\nu} \; \frac{N(\lambda)}{M'(\lambda) \; + \; \frac{\alpha}{c\nu} N'(\lambda)} \bigg|_{\lambda = \lambda_k(\alpha)}.$$

In particular

$$\lambda_k'(0) = \frac{-1}{c\nu} \left.\frac{N(\lambda)}{M'(\lambda)}\right|_{\lambda=\lambda_k(0)} .$$

We can show that

$$\begin{aligned} \frac{d}{d\alpha}(\operatorname{Re}\lambda_k(\alpha)) &= \frac{-\mu_k^2}{L^2\nu}\left(\frac{2}{Lc\nu}\right), \quad \alpha = 0 \\ &= \frac{c}{2\alpha^2 L}, \qquad \alpha = +\infty \end{aligned} \tag{14}$$

$$\frac{d}{d\alpha}(\operatorname{Im}\lambda_k(\alpha)) \geq 0.$$

A root locus of the first mode is shown in Figure 3. The damping ($= |\operatorname{Re}\lambda_k|$) increases with the gain until a critical value of the gain is reached and thereafter decreases to zero. Note that by virtue of (14) we have actually "proportional damping" for small gain. A plot of the critical value of the gain versus the mode number is given in Figure 4.

Resolvent

Let us now return to the resolvent — or solving (3). We note that

$$g(\lambda, s) = \frac{1}{2\gamma^3}\int_0^s (\operatorname{Sinh}\gamma(s-\sigma) - \operatorname{Sin}\gamma(s-\sigma))\,\nu^2(h_2(\sigma) + \lambda h_1(\sigma))\,d\sigma$$

is a "particular" solution of

$$\lambda^2\nu^2 f_1 + f_1'''' = \nu^2(h_2 + \lambda h_1)$$

such that

$$f_1(0) = f_1'(0) = 0.$$

Hence we can express the solution $f_1(\lambda, s)$, where we have included λ to indicate the dependence on λ, as:

$$f_1(\lambda, s) = g(\lambda, s) + a(\lambda)(\operatorname{Cosh}\gamma s - \operatorname{Cos}\gamma s) + b(\lambda)(\operatorname{Sinh}\gamma s - \operatorname{Sin}\gamma s), \quad 0 < s < L$$

where the coefficients $a(\lambda)$, $b(\lambda)$ are determined from

$$\begin{vmatrix} a(\lambda) \\ b(\lambda) \end{vmatrix} = H(\lambda)^{-1}\begin{vmatrix} \alpha b - \alpha\lambda g'(\lambda, L) - cg''(\lambda, L) \\ -g'''(\lambda, L) \end{vmatrix}$$

where the primes again denote derivatives with respect to the variable s. Hence letting

$$H(\lambda) = \begin{vmatrix} h_{11}(\lambda) & h_{12}(\lambda) \\ h_{21}(\lambda) & h_{22}(\lambda) \end{vmatrix}$$

and defining

$$\tilde{H}(\lambda) = \begin{vmatrix} h_{22}(\lambda) & -h_{12}(\lambda) \\ -h_{21}(\lambda) & h_{11}(\lambda) \end{vmatrix}$$

so that

$$H(\lambda)\tilde{H}(\lambda) = D(\lambda)I = \tilde{H}(\lambda)H(\lambda),$$

$$a(\lambda) = \frac{1}{D(\lambda)}[h_{22}(\lambda)(\alpha b - \alpha\lambda g'(\lambda, L) - cg''(\lambda, L)) + h_{12}(\lambda)g'''(\lambda, L)]$$

$$b(\lambda) = \frac{1}{D(\lambda)}[-h_{21}(\lambda)(\alpha b - \alpha\lambda g'(\lambda, L) - cg''(\lambda, L)) - h_{11}(\lambda)g'''(\lambda, L)].$$

We can cast the Green's function in the form:

$$\begin{aligned} f_1(\lambda, s) &= \int_0^s \frac{K(\lambda; s, \sigma)}{D(\lambda)}\, h(\sigma)\, d\sigma \;+\; \frac{\int_s^L K(\lambda; \sigma, s)}{D(\lambda)}\, h(\sigma)\, d\sigma \\ &\quad + \frac{\alpha b}{D(\lambda)} \left[h_{22}(\lambda)(\operatorname{Cosh}\gamma s - \operatorname{Cos}\gamma s) \;-\; h_{21}(\lambda)(\operatorname{Sinh}\gamma s - \operatorname{Sin}\gamma s) \right] \end{aligned}$$

$$\begin{aligned} K(\lambda; \sigma, s) \;=\; &(\operatorname{Cosh}\gamma s - \operatorname{Cos}\gamma s) \Bigg[\left(2h_{12} - \frac{2\alpha\lambda}{\gamma^2} h_{22}\right) \operatorname{Cosh}\gamma(L - \sigma) \;+\; \left(2h_{12} + \frac{2\alpha\lambda}{\gamma^2} h_{22}\right) \operatorname{Cos}\gamma(L - \sigma) \\ &\qquad - \frac{2ch_{22}}{\gamma} \left(\operatorname{Sinh}\gamma(L - \sigma) \;+\; \operatorname{Sin}\gamma(L - \sigma)\right) \Bigg] \\ &+ (\operatorname{Sinh}\gamma s - \operatorname{Sin}\gamma s) \Bigg[\left(\frac{2\alpha\lambda h_{21}}{\gamma^2} - 2h_{11}\right) \operatorname{Cosh}\gamma(L - \sigma) \;+\; \left(\frac{-2\alpha\lambda h_{21}}{\gamma^2} - 2h_{11}\right) \operatorname{Cos}\gamma(L - \sigma) \\ &\qquad + \frac{2ch_{21}}{\gamma} \left(\operatorname{Sinh}\gamma(L - \sigma) \;+\; \operatorname{Sin}\gamma(L - \sigma)\right) \Bigg], \quad s < \sigma \end{aligned} \tag{15}$$

$$h = m(h_2 + \lambda h_1)$$

$$D(\lambda) \;=\; -2\lambda^2 \nu^2 \left[c\gamma(1 + \operatorname{Cosh}\gamma L \operatorname{Cos}\gamma L) \;+\; \lambda\alpha\, (\operatorname{Sinh}\gamma L \operatorname{Cos}\gamma L \;+\; \operatorname{Cosh}\gamma L \operatorname{Sin}\gamma L) \right].$$

Finally

$$\mathcal{R}(\lambda, \mathcal{A}) \begin{vmatrix} h_1 \\ b \\ h_2 \end{vmatrix} = \begin{vmatrix} f_1(\lambda, s) \\ f_1'(\lambda, 0) \\ \lambda f_1(\lambda, s) - h_1(s) \end{vmatrix}.$$

Note that setting $\alpha = 0$ in (15) we get the Green's function for the clamped/free-free beam. In particular

$$\mathcal{R}(0, \mathcal{A}) \begin{vmatrix} h_1 \\ h_1(0) \\ h_2 \end{vmatrix} = \begin{vmatrix} Kh_2 \\ (Kh_2)(0) \\ -h_1 \end{vmatrix} + \alpha h_1(0) \begin{vmatrix} \frac{L-s}{c} \\ \frac{L}{c} \\ 0 \end{vmatrix}$$

where Kh_2 is the function given by

$$\frac{m}{c} \int_0^s (L - \sigma) h_2(\sigma)\, d\sigma \;+\; \frac{m}{c} \int_L^s (L - \sigma) h_2(\sigma)\, d\sigma, \quad 0 < s < L.$$

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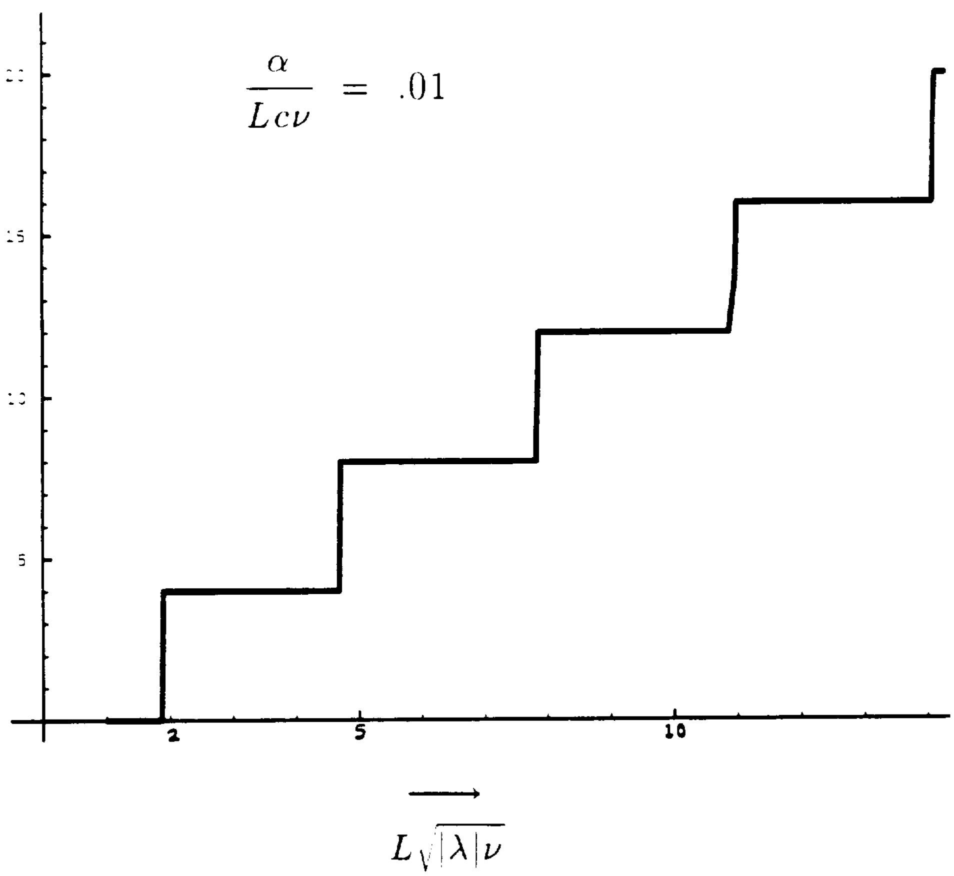

Figure 1: $n(|\lambda|)$ Zeros.

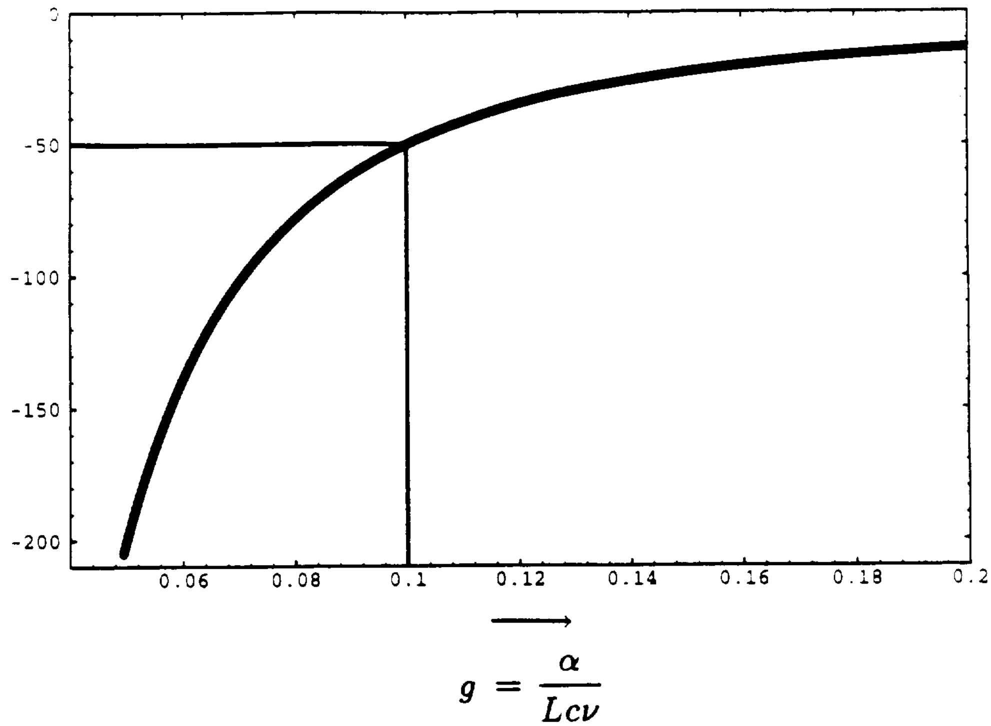

Figure 2: Deadbeat Mode (Real Eigenvalue).

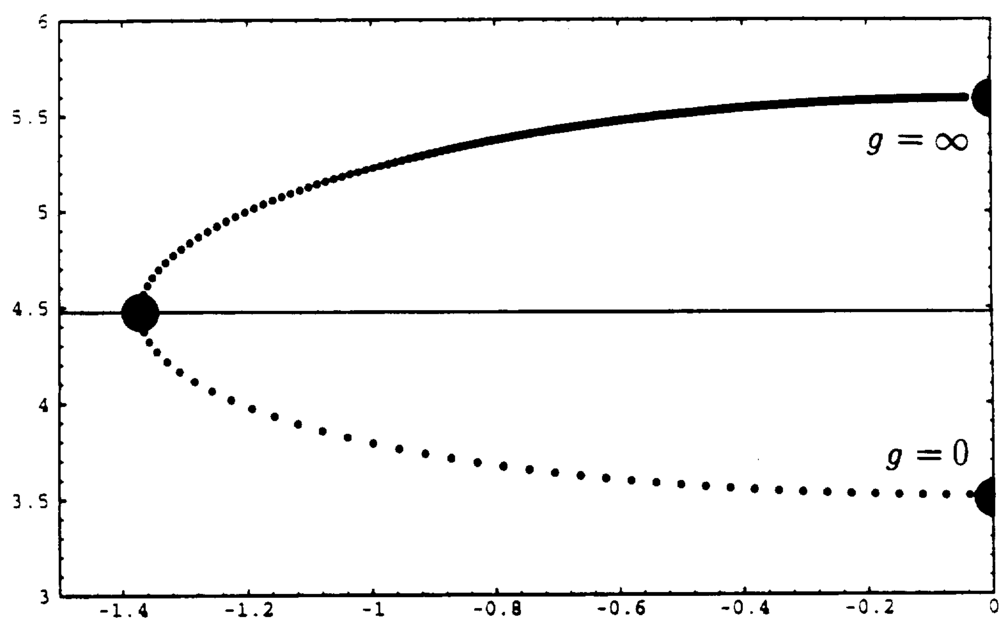

Figure 3: Root Locus: First Mode $\lambda = \sigma + i\omega$.

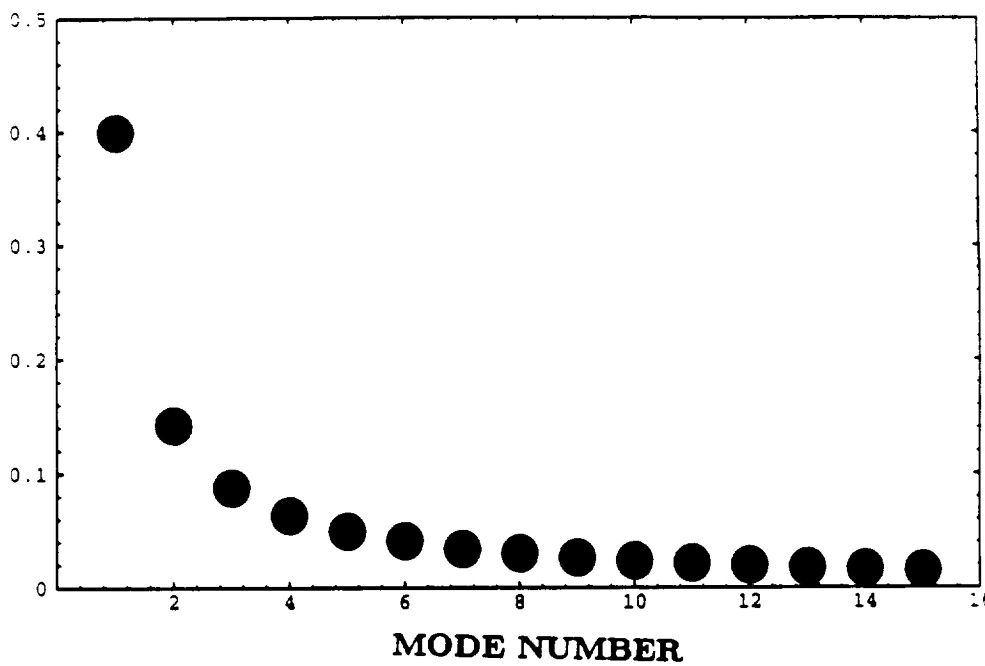

Figure 4: Critical Gain vs. Mode.